# Lecture 18. Matrices and Linear Systems

## Review of Matrix Notation and Terminology

An $m \times n$ matrix $\mathbf{A}$ is a rectangular array of $mn$ numbers (or elements) arranged in $m$ (horizontal) rows and $n$ (vertical) columns:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots a_{1n} \\ a_{21} & a_{22} & \cdots a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots a_{mn} \end{pmatrix}$$

Two $m \times n$ matrices $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ are said to be equal if corresponding elements are equal. We have

$$\mathbf{A} + \mathbf{B} = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

$$c\mathbf{A} = \mathbf{A}c = [ca_{ij}]$$

We have

- $A + B = B + A$
- $(A+B) + C = A + (B+C)$
- $c(A+B) = cA + cB$
- $(c+d) \cdot A = cA + dA$

The **transpose** $\mathbf{A}^T$ of the $m \times n$ matrix $\mathbf{A} = [a_{ij}]$ is the $n \times m$ matrix whose $j$ th column is the $j$ th row of $\mathbf{A}$

Example : $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}_{3\times 2}$ then $A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}_{2\times 3}$

## Matrix Multiplication

If

$$\mathbf{a} = \begin{bmatrix} a_1 \, a_2 \, \cdots \, a_p \end{bmatrix}_{1\times p} \text{ and } \mathbf{b} = \begin{bmatrix} b_1 \, b_2 \, \cdots \, b_p \end{bmatrix}^T_{p\times 1}$$

then the **scalar product** $\mathbf{a} \cdot \mathbf{b}$ is defined as follows:

$$\mathbf{a} \cdot \mathbf{b} = \sum_{k=1}^{p} a_k b_k = a_1 b_1 + a_2 b_2 + \cdots + a_p b_p$$

The product $\mathbf{AB}$ of two matrices is defined only if the number of columns of $\mathbf{A}$ is equal to the number of rows of $\mathbf{B}$. If $\mathbf{A}$ is an $m \times p$ matrix and $\mathbf{B}$ is a $p \times n$ matrix, then their product $\mathbf{AB}$ is the $m \times n$ matrix

$$\mathbf{C} = \begin{bmatrix} c_{ij} \end{bmatrix}$$

where $c_{ij}$ is the scalar product of the $i$th row vector $\mathbf{a}_i$ of $\mathbf{A}$ and the $j$th column vector $\mathbf{b}_j$ of $\mathbf{B}$. Thus

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} \mathbf{a}_i \cdot \mathbf{b}_j \end{bmatrix}$$

If $\mathbf{A} = \begin{bmatrix} a_{ij} \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b_{ij} \end{bmatrix}$, then we have

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

For the computation by hand, it is easy to remember by visualizing the picture

$$\begin{array}{c} \\ \\ \\ \mathbf{a}_i \longrightarrow \\ \\ \\ \\ \end{array} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ip} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \begin{array}{c} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pj} & \cdots & b_{pn} \end{bmatrix} \\ \begin{array}{cccccc} \phantom{b_{11}} & \phantom{b_{12}} & \phantom{\cdots} & \uparrow & \phantom{\cdots} & \phantom{b_{1n}} \\ & & & \mathbf{b}_j & & \end{array} \end{array},$$

which shows that one forms the dot product of the row vector $\mathbf{a}_i$ with the column vector $\mathbf{b}_j$ to obtain the element $c_{ij}$ in the $i$th row and the $j$ th column of $\mathbf{AB}$.

## Inverse Matrices

$a \in \mathbb{R}$, $a \cdot 1 = 1 \cdot a = a$

$a \cdot b = b \cdot a = 1$

then we call $b$ an inverse of $a$, which is $\frac{1}{a} = a^{-1}$.

The **identity** matrix of order $n$ is the square matrix

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

We have

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A}$$

If $\mathbf{A}$ is a square matrix, then an inverse of $\mathbf{A}$ is a square matrix $\mathbf{B}$ of the same order as $\mathbf{A}$ such that both

$$\mathbf{AB} = \mathbf{I} \text{ and } \mathbf{BA} = \mathbf{I}$$

We denote such $\mathbf{B}$ by $\mathbf{A}^{-1}$.

Rmk: Note $B$ may not exist!

Eg: $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

In linear algebra it is proved that $\mathbf{A}^{-1}$exists if and only if the determinant $\det(\mathbf{A})$ of the square matrix $\mathbf{A}$ is nonzero. If so, the matrix $\mathbf{A}$ is said to be **nonsingular**; if $\det(\mathbf{A}) = 0$, then $\mathbf{A}$ is called a **singular** matrix.

**Example 1** Find $\mathbf{AB}$ and $\mathbf{BA}$ given

$$\mathbf{A} = \begin{pmatrix} 5 & 3 & 4 \\ 3 & -2 & 1 \end{pmatrix}_{2\times 3} \text{ and } \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 3 \end{pmatrix}_{3\times 2}$$

ANS:

$$AB = \begin{pmatrix} 5 & 3 & 4 \\ 3 & -2 & 1 \end{pmatrix}_{2\times 3} \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 3 \end{pmatrix}_{3\times 2} = \begin{pmatrix} 5\times 1 + 3\times 4 + 4\times 2 & 5\times 2 + 3\times 5 + 4\times 3 \\ 3\times 1 - 2\times 2 + 3 & 3\times 2 - 2\times 5 + 3 \end{pmatrix}$$

$$= \begin{pmatrix} 25 & 37 \\ -3 & -1 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 3 \end{pmatrix}_{3\times 2} \begin{pmatrix} 5 & 3 & 4 \\ 3 & -2 & 1 \end{pmatrix}_{2\times 3} = \begin{pmatrix} 5+6 & 3-4 & 4+2 \\ 20+15 & 12-10 & 16+5 \\ 10+9 & 0 & 8+3 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & -1 & 6 \\ 25 & 2 & 21 \\ 19 & 0 & 11 \end{pmatrix}_{3\times 3}$$

Note $AB \neq BA$

## Matrix-Valued Functions

A **matrix-valued function** is a matrix such as

$$\mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots a_{2n}(t) \\ \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots a_{nn}(t) \end{pmatrix}$$

in which each entry is a function of $t$.

We say that the matrix function $\mathbf{A}(t)$ is **continuous** (or **differentiable**) at a point (or on an interval) if each of its elements has the same property. The **derivative** of a differentiable matrix function is defined by elementwise differentiation:

$$\mathbf{A}'(t) = \frac{d\mathbf{A}}{dt} = [\frac{da_{ij}}{dt}]$$

**Example 2** Let $A$ and $B$ be the matrices as in Example 1. Let

$$\mathbf{x} = \begin{pmatrix} e^{-2t} \\ 3t \end{pmatrix} \quad \text{and} \quad \mathbf{y} = \begin{pmatrix} t^3 \\ \tan t \\ \sin t \end{pmatrix}$$

Find $\mathbf{Ay}$ and $\mathbf{Bx}$. Are the products $\mathbf{Ax}$ and $\mathbf{By}$ well-defined?

ANS: $A\vec{y} = \begin{pmatrix} 5 & 3 & 4 \\ 3 & -2 & 1 \end{pmatrix}_{2\times3} \begin{pmatrix} t^3 \\ \tan t \\ \sin t \end{pmatrix}_{3\times1} = \begin{pmatrix} 5t^3 + 3\tan t + 4\sin t \\ 3t^3 - 2\tan t + \sin t \end{pmatrix}_{2\times1}$

$B\vec{x} = \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 3 \end{pmatrix}_{3\times2} \begin{pmatrix} e^{-2t} \\ 3t \end{pmatrix}_{2\times1} = \begin{pmatrix} e^{-2t} + 6t \\ 4e^{-2t} + 15t \\ 2e^{-2t} + 9t \end{pmatrix}_{3\times1}$

The products $A\vec{x}$ and $B\vec{y}$ are not well-defined.

Since $A$ is a $2\times\underline{3}$ matrix but $\vec{x}$ is a $\underline{2}\times1$ matrix.

does not equal.

Also $B$ is a $3\times\underline{2}$ matrix but $\vec{y}$ is a $\underline{3}\times1$ matrix

does not equal

**Example 3** Find $\mathbf{A}'$ if

$$\mathbf{A}(t) = \begin{pmatrix} 3t & t^2 \\ t^3 & 3+t^4 \end{pmatrix}$$

ANS: $A'(t) = \begin{bmatrix} (3t)' & (t^2)' \\ (t^3)' & (3+t^4)' \end{bmatrix} = \begin{bmatrix} 3 & 2t \\ 3t^2 & 4t^3 \end{bmatrix}$

## First-Order Linear Systems

We discuss here the general system of n first-order linear equations

$$\begin{aligned} x_1' &= p_{11}x_1 + p_{12}x_2 + \cdots p_{1n}x_n + f_1(t), \\ x_2' &= p_{21}x_1 + p_{22}x_2 + \cdots p_{2n}x_n + f_2(t), \\ x_3' &= p_{31}x_1 + p_{32}x_3 + \cdots p_{3n}x_n + f_3(t), \\ &\vdots \\ x_n' &= p_{n1}x_1 + p_{n2}x_2 + \cdots p_{nn}x_n + f_n(t), \end{aligned}$$

$$\Longleftrightarrow \begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ p_{31} & p_{32} & \cdots & p_{3n} \\ \vdots & \vdots & \vdots & \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

If we introduce the coefficient matrix

$$\mathbf{P}(t) = \left[ p_{ij}(t) \right]$$

and the column vectors

$$\mathbf{x} = \left[ x_i \right] \quad \text{and} \quad \mathbf{f}(t) = \left[ f_i(t) \right]$$

Then the above system takes the form of a single matrix equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t) \tag{1}$$

A solution of Eq. (1) on the open interval $I$ is a column vector function $\mathbf{x}(t) = \left[ x_i(t) \right]$ such that the component functions of $\mathbf{x}$ satisfy the above system identically on $I$.

**Example 4** Write the given system in the for $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t)$.

(1) $$\begin{aligned} x' &= x + 3y + 2e^t, \\ y' &= 4x - y - t^2 \end{aligned}$$

ANS: We have $\vec{x} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, $P(t) = \begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix}$, $\vec{f}(t) = \begin{bmatrix} 2e^t \\ -t^2 \end{bmatrix}$

Then
$$\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}'}_{\vec{x}'(t)} = \underbrace{\begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix}}_{P(t)} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}(t)} + \underbrace{\begin{bmatrix} 2e^t \\ -t^2 \end{bmatrix}}_{\vec{f}(t)}$$

(2) $x' = 2x - 3y$, $+ 0 \cdot z$

$y' = x + y + 2z$,

$z' = 5y - 7z$ $+ 0 \cdot x$

We have $\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$, $P(t) = \begin{bmatrix} 2 & -3 & 0 \\ 1 & 1 & 2 \\ 0 & 5 & -7 \end{bmatrix}$, $\vec{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

So
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \begin{bmatrix} 2 & -3 & 0 \\ 1 & 1 & 2 \\ 0 & 5 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

To solve the Eq. (1) in general, we consider first the the **associated homogeneous equation**

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}(t)\mathbf{x} \tag{2}$$

We expect it to have $n$ solutions $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ that are independent in some appropriate sense, and such that every solution of Eq. (2) is a linear combination of these $n$ particular solutions. We will talk about the structure of the solutions in **Lecture 19.**

Now consider the single $n$th-order equation

$$x^{(n)} = f(t, x, x', \cdots, x^{(n-1)})$$

It is of both practical and theoretical importance that any such higher-order equation can be transformed into an equivalent system of first-order equations.

We introduce the independent variables $x_1, x_2, \cdots, x_n$ as follows:

$$x_1 = x,\ x_2 = x',\ x_3 = x'',\ \cdots, x_n = x^{(n-1)}.$$

Then we have the following system

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \cdots \\ x_{n-1}' = x_n \\ x_n' = f(t, x_1, x_2, \cdots, x_n) \end{cases}$$

**Example 5** Transform the given differential equation into an equivalent system of first-order differential equations.

$$x'' + 2x' + 26x = 34\cos 4t \Rightarrow x'' = -2x' - 26x + 34\cos 4t$$

ANS: Let $\underline{x_1} = x$ , $\underline{x_2} = x'$

Then $x_1' = x' = x_2 = \underline{0} \cdot x_1 + \underline{1} \cdot x_2 + 0$

$$x_2' = (x')' = x'' = -2\cancel{x'}^{\,x_2} - 26\cancel{x}^{\,x_1} + 34\cos 4t$$

$$= -2x_2 - 26x_1 + 34\cos 4t$$

$$= \underline{-26}\,x_1 \underline{-2}\,x_2 + 34\cos 4t$$

So $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -26 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 34\cos 4t \end{bmatrix}$$

**Exercise 6.** Consider the system of higher order differential equations

$$y'' = t^{-1}y' + 4y - tz + (\sin t)z' + e^{5t},$$
$$z'' = y - 4z'.$$

Rewrite the given system of two second order differential equations as a system of four first order linear differential equations of the form $\vec{\boldsymbol{y}}\,' = P(t)\vec{\boldsymbol{y}} + \vec{\boldsymbol{g}}(t)$. Use the following change of variables

$$\vec{\boldsymbol{y}}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ y'(t) \\ z(t) \\ z'(t) \end{bmatrix}.$$

ANS: Using the suggested change of variables, we have

$$y_1 = y, \quad y_2 = y_1', \quad y_3 = z, \quad y_4 = z'$$

In order to write the eqn as $\vec{y}\,' = P(t)\vec{y} + \vec{g}(t)$.

We need to figure out the eqns with left hand side

as $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}'$ and right hand side only in terms of $y_1, y_2, y_3, y_4$

(change of variables)

Thus $y_1' = y_2 = 0 \cdot y_1 + 1 \cdot y_2 + 0 \cdot y_3 + 0 \cdot y_4$

$$y_2' = t^{-1}y_2 + 4y_1 - t\,y_3 + (\sin t)\,y_4 + e^{5t}$$
$$= 4y_1 + t^{-1}y_2 - ty_3 + (\sin t)\,y_4 + e^{5t}$$

$$y_3' = z' = y_4 = 0 \cdot y_1 + 0 \cdot y_2 + 0 \cdot y_3 + 1 \cdot y_4$$

$$y_4' = z'' = y_1 - 4y_4 = 1 \cdot y_1 + 0 \cdot y_2 + 0 \cdot y_3 - 4y_4$$

Thus
$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 4 & t^{-1} & -t & \sin t \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} + \begin{bmatrix} 0 \\ e^{5t} \\ 0 \\ 0 \end{bmatrix}$$

So
$$\vec{y}'(t) = P(t)\,\vec{y}(t) + \vec{g}(t)$$

where
$$P(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 4 & t^{-1} & -t & \sin t \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -4 \end{bmatrix} \quad \text{and} \quad \vec{g}(t) = \begin{bmatrix} 0 \\ e^{5t} \\ 0 \\ 0 \end{bmatrix}$$